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Time Domain Analysis and Synthesis of Notch Filters

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The response of notch filters to sudden excitations is analyzed. Unit step and stepped trigonometric inputs are considered for the class of filters derived from low-pass networks by a frequency transformation. It is possible in some cases to approximate the transient solutions in terms of Laguerre functions and deduce general properties of notch filters from these solutions. The use of phasing sections to modify the transient response is also examined. It is shown that this method can be used to effectively reduce the overshoot in the response to a stepped trigonometric excitation.

I. INTRODUCTION

In order to accurately measure noise levels in compandored communication systems, it is necessary to set the compandor characteristics at approximately the values associated with signal transmission. This is done by applying a so-called "holding tone" which is subsequently removed by a notch filter incorporated into the measuring set. This investigation originated in connection with the design of such a notch filter for an impulse noise counter.[1] The filter has to meet both frequency and time domain requirements. The frequency response requirements could be readily met with existing filter design procedures. However, the time domain characteristics of notch filters needed investigation with regard to the suitability for the present application. The time domain requirement is that the filter when combined with a C-message weighting filter[2] and excited with a stepped trigonometric time function at the notch frequency should, in the transient state, have only a specified overshoot level. This requirement is imposed by the necessity to distinguish in the measuring set between sudden gain and phase variations and impulse noise.

To examine the transient response of notch filters, a class of such filters derived from low-pass filters by a frequency transformation is considered. The transient response to a unit step function, and to a stepped trigonometric function with the notch frequency, is expressed in terms of the low-pass impulse response. It is shown that the low-pass and notch filter response are for both excitations related by a Hankel transform. Some general properties of the transient response are deduced by considering notch filters for which the response can be obtained approximately in terms of generalized Laguerre functions.

This investigation shows that notch filters would distort narrow time pulses of duration less than one-half the notch frequency period. The amount of distortion is related to the notch depth. The response of a notch filter to a stepped trigonometric function can be kept to a low level only after a certain time duration, which depends on the filter parameters. The transient response of a notch filter followed by a low-pass filter may still assume large values at relatively short times from the beginning of the response. A method of decreasing the transient response at such times by the use of phasing sections is also presented. The use of phasing sections is of particular importance where it is necessary to modify the transient response without affecting the frequency response.

Although this work is primarily concerned with notch filters, the methods used may also be applied to determine the transient response of high-pass and bandpass filters, when derived from low-pass filters by a frequency transformation.

II. TRANSIENT RESPONSE OF NOTCH FILTERS TO A UNIT STEP FUNCTION

A class of notch filters derived from low-pass filters by a frequency transformation[3] is considered. The transformation corresponds to replacing the inductances and capacitances in the low-pass filters with parallel and series resonant circuits, respectively. Let $T_L(s)$ be the low-pass transfer function in the complex frequency domain s. The transformation is given by

$$s = \frac{\beta z}{z^2 + \omega_o^2}, \tag{1}$$

where $\omega_o = 2\pi f_o$, f_o = notch frequency, and β is a constant. (For low-pass filters normalized such that for $s = j\omega$ the 3-dB bandwidth is at $\omega = 1$, β is the 3-dB circular bandwidth of the notch filter.) The transfer function of the notch filter in the complex frequency domain z, $T_N(z)$, is related to the low-pass transfer function $T_L(s)$ by

$$T_N(z) = T_L\left(\frac{\beta z}{z^2 + \omega_o^2}\right). \tag{2}$$

To investigate the transient response of notch filters, that response is related to the impulse response of the low-pass filter. Such a relationship is obtained by first expressing the transfer function of the low-pass filter in terms of the Laplace transform of the impulse response, $f_L(t)$,

$$T_L(s) = \int_0^\infty e^{-st} f_L(t) dt; \tag{3}$$

then from (2) the transfer function of the notch filter by

$$T_N(z) = \int_0^\infty \exp - \left[\frac{\beta z t}{z^2 + \omega_o^2} \right] f_L(t) dt. \tag{4}$$

The time response, $f_{NS}(t)$, of the notch filter to a unit step function can now be obtained by inversion of the Laplace transformation, through integration in the complex plane over the contour Γ, as follows:

$$f_{NS}(t) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} e^{zt} \frac{T_N(z)}{z} dz, \tag{5}$$

where γ is a positive constant. The relationship between step response of the notch filter and the impulse response of the low-pass filter is obtained by the substitution of (4) into (5) yielding

$$f_{NS} = \frac{1}{2\pi j} \int_\Gamma \frac{e^{zt}}{z} \int_0^\infty \exp - \left(\frac{\beta z u}{z^2 + \omega_o^2} \right) f_L(u) du\, dz. \tag{6}$$

In Appendix A it is shown that (6) can be expressed as follows:

$$f_{NS}(t) = \left[\int_0^\infty f_L(u) du \right] \cdot 1(t) - \int_0^t \left[\int_0^\infty f_L(u) \sqrt{\frac{\beta u}{x}} J_1(2\sqrt{\beta x u}) du \right] J_o(2\omega_o \sqrt{xt - x^2}) dx, \tag{7}$$

where $1(t)$ is the unit step function and $J_n(y)$ is a Bessel function of order n.

In (7) the first term is the undistorted unit step response and the second term expresses the distortion introduced by the notch filter.

The first term can be simplified by observing that from (3)

$$\int_0^\infty f_L(t) dt = T_L(0). \tag{8}$$

Without loss of generality, $T_L(s)$ can be normalized to be equal to unity at zero frequency. The expression in the brackets of the second term contains a Hankel transform,[4] of order one, hence (7) can be written

$$f_{NS}(t) = 1(t) - 2\int_0^t \psi_1(2\sqrt{\beta x})\sqrt{\frac{\beta}{x}}\,J_o(2\omega_o\sqrt{xt - x^2})dx, \qquad (9)$$

where ψ_1 is the Hankel transform.

The integration in (9) can be performed approximately by using the method of stationary phase. This is a good approximation when the Hankel transform is a slowly varying function in comparison to the Bessel function. It is also shown in Appendix A that with the stationary phase approximation (9) can be approximated by

$$f_{NS}(t) = \left\{1.0 - \frac{\sin\omega_o t}{\omega_o}\int_0^\infty f_L(u)\sqrt{\frac{2\beta u}{t}}\,J_1(\sqrt{2\beta ut})du\right\}\cdot 1(t). \qquad (10)$$

An example considered subsequently shows that (10) is indeed a good approximation for $\beta \ll \omega_o$, i.e., for narrowband notch filters.

III. TRANSIENT RESPONSE OF NOTCH FILTERS TO A UNIT STEPPED SINE FUNCTION

The transient response to this function is obtained in a manner similar to the response to a unit step function. However, a more general response function, $f_{Nm}(t)$, is considered,

$$f_{Nm}(t) = \frac{\omega_o^{2m+1}}{2\pi j}\int_\Gamma e^{zt}\,\frac{T_L\left(\dfrac{\beta z}{z^2 + \omega_o^2}\right)}{(z^2 + \omega_o^2)^{m+1}}\,dz, \qquad (11)$$

providing for the possibility of a low-pass notch combination. The time response of a notch filter to a stepped sine function with the notch frequency is obtained as a special case, by setting $m = 0$. Proceeding in a manner similar to Appendix A, it can be readily shown that (11) can be expressed as follows:

$$f_{Nm}(t) = \omega_o\int_0^t\left\{\int_0^\infty f_L(u)\left[\frac{\omega_o^2(t-x)}{\beta u}\right]^{m/2} J_m(2\sqrt{\beta ux})du\right\} \cdot J_m[2\omega_o\sqrt{x(t-x)}]dx. \qquad (12)$$

Equation (12) also contains a Hankel transform but of order m. Similarly as before, (12) can be approximated by using the method of stationary phase. It can also be readily shown, from the results in Appendix A, that with this approximation (12) is given by

$$f_{Nm}(t) \approx \sin\left(\omega_o t - \frac{m\pi}{2}\right)\int_0^\infty f_L(u)\left(\frac{\omega_o^2 t}{2\beta u}\right)^{m/2} J_m(\sqrt{2\beta ut})du\cdot 1(t). \qquad (13)$$

It is shown below that, for $m = 0$, (13) is a good approximation for $\beta \ll \omega_o$.

IV. APPLICATION TO PARTICULAR TYPES OF NOTCH FILTERS

4.1 Introduction

To gain some insight into the behavior of the transient response, a low-pass transfer function is chosen for which the integrals (10) and (13) can be evaluated. As mentioned before, these integrals are Hankel transforms. A class of functions for which Hankel transforms can be evaluated in closed form are the generalized Laguerre functions.[5,6] In fact, for these functions the Hankel transforms are self-reciprocal.[5] These functions form orthogonal sets so that, in principle, any passive filter impulse response can be expanded in terms of these functions. However, the impulse response of some low-pass filters may be approximated closely by a single Laguerre function. Such a response can be used to obtain an estimate of the notch filter response.

The low-pass transfer functions considered are of the following form:

$$T_L(s) = \frac{(s/\alpha_2 + 1)^n}{(s/\alpha_1 + 1)^{n+m+1}} \tag{14}$$

with m and n integers, $n, m \geqq 0$, and $\alpha_1 > 0$.

The transfer function (14) is normalized such that $T_L(0) = 1.0$. This function can be considered as consisting of $(m + 1)$ cascaded low-pass filter sections and, for $\alpha_2 = -\alpha_1$, of n phasing sections. For physical realizability of the n cascaded sections, it is necessary and sufficient[7,8] that $|\alpha_2| \geqq \alpha_1$.

The impulse response corresponding to the transfer function (14) is[9]

$$f_L(t) = \frac{n!}{(n+m)!}\alpha_1\left(\frac{\alpha_1}{\alpha_2}\right)^n(\alpha_1 t)^m e^{-\alpha_1 t} L_n^m[(\alpha_1 - \alpha_2)t], \tag{15}$$

where $L_n^m(x)$ are generalized Laguerre polynomials given by[10]

$$L_n^m(x) = \sum_{k=0}^{n}(-1)^k\binom{n+m}{n-k}\frac{x^k}{k!}. \tag{16}$$

These polynomials are oscillatory for positive arguments and monotonically increasing for negative arguments.

4.2 Unit step response

The integral (10) with $f_L(u)$ given by (15) can be reduced to a tabulated integral for the special case $n = 0$. The applicable integral is of the form[11]

$$\int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{Y})dx = \frac{n!}{2} e^{-Y} Y^{1/2\mu} L_n^\mu(Y). \tag{17}$$

For this special case the low-pass transfer function is

$$T_L(s) = \frac{1}{(s+1)^{m+1}}, \tag{18}$$

where without loss of generality α_1 is set equal to one.

The transient response of the notch filter is obtained by using (10), (15), and (17) and is approximately

$$f_{NS}(t) \approx \left[1.0 - \frac{\beta}{\omega_o} e^{-(\beta/2)t} L_m^1\left(\frac{\beta}{2}t\right) \sin \omega_o t\right] \cdot 1(t). \tag{19}$$

It is of interest to determine the accuracy of (19) for different values of m. This is done in Appendix B for $m = 0$, 1, and 2 where (19) is compared with the exact solutions. It is shown that the accuracy of (19) is of order β/ω_o when expressions multiplied by both $\sin \omega t$ and $\cos \omega t$ are considered. However, expressions multiplied by $\sin \omega t$ only are of order $(\beta/\omega_o)^2$, so that near the maximum amplitudes of the second term in (19), expected near $\cos \omega_o t \approx 0$, the approximation can be said to be of order $(\beta/\omega_o)^2$. Equation (19) is therefore a very

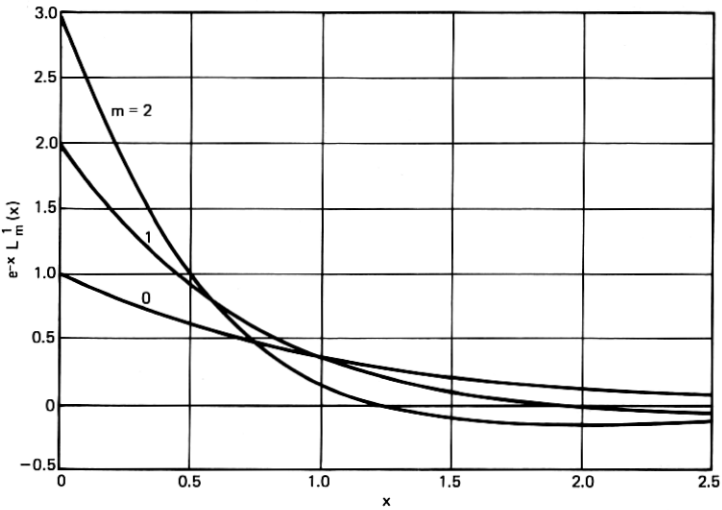

Fig. 1—Laguerre functions, $e^{-x}L_m^1(x)$.

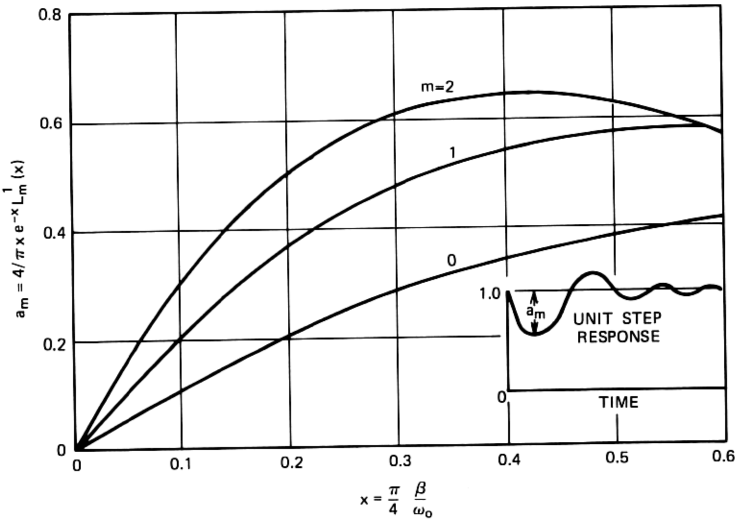

Fig. 2—Values of $a_m(x)$.

good approximation for $\beta/\omega_o \ll 1$, i.e., for notch filters in which the 3-dB bandwidth is much smaller than the notch frequency.

In Fig. 1 the function $e^{-x}L_m^1(x)$ is shown for $m = 0, 1, 2$. It follows from this figure, in conjunction with (19), that the distortion of the unit step function will be particularly pronounced in the time vicinity $T = \pi/2\omega_o$. In Fig. 2 the values of the second term in (19) are plotted as a function of $\beta T/2$ for $m = 0, 1, 2$. This graph gives an estimate of the distortion of the unit step function, and is of particular significance when considering the transient response of notch filters to pulses of short time duration. This investigation shows that pulses of duration less than one-half notch frequency period will be considerably distorted by notch filters.

To obtain a numerical estimate of the distortion, a notch filter with the following requirements is considered: (*i*) Notch frequency—1010 Hz. (*ii*) Notch depth at least -30 dB in the frequency range 995–1025 Hz. A notch filter with these requirements is derived from the low-pass transfer function (18) for $m = 0, 1, 2, 3$.

It readily follows from the transformation (1) that β can be determined from the following equations.

$$\beta = \omega_A 2\pi(f_o - f_1)\left(1 + \frac{f_o}{f_1}\right), \tag{20}$$

where f_o = notch frequency, f_1 is the lower frequency where a notch depth A[dB] is required, and, from (18),

$$\omega_A = \sqrt{\exp - \left(\frac{A}{10(m+1)} \ln 10 \right) - 1}. \tag{21}$$

The 3-dB bandwidth B of the notch, also obtained from (1), is

$$B = \frac{\beta}{2\pi\omega_B}, \tag{22}$$

where ω_B is given by (21) with $A = -3.0$.

Table I lists the filter parameters, including the maximum value, a_m, of the second term in (19).

It is evident from Table I that the 3-dB bandwidth and the distortion term a_m decrease as the number of sections increases. The difference is particularly pronounced between $m = 0$ and $m = 1$. A further decrease in the distortion term can be obtained by decreasing β; however, this also reduces the depth of the notch.

The listed values of a_m seem to be representative of what is obtainable with notch filters of specified notch depth and 3-dB bandwidth. For example, computation of a two-section notch filter derived from a Tschebyscheff low-pass filter with 0.5-dB ripple gave a value for a_m of 0.18. The notch depth of this filter was also -30 dB and the 3-dB bandwidth 170 Hz. The lower value obtained with this filter can be attributed to the more oscillatory behavior[12] of the low-pass impulse response. Equation (10) suggests such an interpretation.

4.3 Transient response to a stepped sine function

The low-pass transfer function (14) and the corresponding impulse response (15) are again considered. With that impulse response the integral (13) is given by

$$f_{Nm}(t) = \frac{\alpha_1^{n+m+1}}{\alpha_2^n} \frac{n!}{(n+m)!} \left(\frac{\omega_o t}{2\beta} \right)^{m/2} \sin \left(\omega_o t - \frac{m\pi}{2} \right) \cdot \int_0^\infty e^{-\alpha_1 u} u^{m/2} L_n^m [(\alpha_1 - \alpha_2)u] J_m(\sqrt{2\beta u t}) du. \tag{23}$$

The integral (23) can be evaluated in closed form, again yielding Laguerre functions[6]

$$f_{Nm}(t) = \frac{n!}{(n+m)!} \left(\frac{\omega_o t}{2} \right)^m e^{-\beta t/2\alpha_1} L_n^m \left[\frac{\beta t}{2} \frac{(\alpha_2 - \alpha_1)}{\alpha_1 \alpha_2} \right] \cdot \sin \left(\omega_o t - \frac{m\pi}{2} \right). \tag{24}$$

Table I—Notch filter parameters

m	ω_A	ω_B	β[kHz]	B[Hz]	a_m
0	31.61	1	6.00	955	0.527*
1	5.53	0.64	1.05	260	0.273
2	3.00	0.51	0.57	178	0.234
3	2.15	0.43	0.41	149	0.227

* Computed from the exact expression, and occurs at a time, $t = 0.12$ ms.

It is noted that with the condition for physical realizability of the n cascaded sections in (14), $|\alpha_2| \geqq \alpha_1$, that the argument of the Laguerre functions is always positive, and hence the functions are oscillatory.

From (11), (14), and (1), (24) corresponds to the inverse of the following Laplace transform:

$$T(z) = \left[\frac{\omega_o}{z^2 + \omega_o^2}\right]\left[\frac{\omega_o^{2m}}{(z^2 + \omega_o^2 + \beta/\alpha_1 z)^m}\right] \cdot \left[\frac{z^2 + \omega_o^2}{z^2 + \omega_o^2 + \beta/\alpha_1 z}\right]\left[\left(\frac{z^2 + \omega_o^2 + \beta/\alpha_2 z}{z^2 + \omega_o^2 + \beta/\alpha_1 z}\right)^n\right]. \tag{25}$$

The terms in the brackets in (25) can be interpreted as transforms of (*i*) a stepped sine function, (*ii*) an m-section low-pass filter, (*iii*) a notch filter section, (*iv*) n phasing sections for $\alpha_2 = -\alpha_1$ or n additional notch sections for $\alpha_2 \to \infty$.

The transient response of a notch filter followed by a low-pass filter is of interest. However, (25) is restricted to a particular low-pass filter with a high-frequency cutoff in the vicinity of the notch frequency, and will not be considered further.

Setting $m = 0$, (24) simplifies to

$$f_{N0}(t) = e^{-(\beta/2\alpha_1)t} L_n^0\left[\frac{\beta t}{2\alpha_1}\left(1 - \frac{\alpha_1}{\alpha_2}\right)\right] \sin \omega_o t. \tag{26}$$

The special case $\alpha_2 \to \infty$, treated previously for the unit step response, can be obtained from (26) and is in agreement with the result obtained by performing the integration directly by using (17).

For $\alpha_2 \to \infty$ and $m = 0, 1, 2$, a comparison was made between the exact solutions and the approximate solution (26). The comparison showed the same accuracies as for the unit step response.

The time response due to a stepped cosine excitation can be obtained by differentiating (26) and dividing by ω_o. It readily follows that, within the accuracy of (26), the same expression is obtained but with sin $\omega_o t$ replaced by cos $\omega_o t$.

It is noted that (26) gives the correct value for $t = 0$. This value can be obtained by using the initial value theorem of Laplace transforms.[13]

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t). \tag{27}$$

Graphs of $e^{-x}L_n^0(x)$ are shown in Fig. 3 for $n = 0, 1, 2$. These graphs give the envelope of the transient response (26) for $\alpha_2 \to \infty$. The effect of finite values of α_2 can be deduced from the graphs. For example, for α_2 negative the arguments of the Laguerre functions increase reaching maximum values of $(\beta t)/\alpha_1$ for $\alpha_1 = -\alpha_2$. Therefore, for the same βt the spacing between the zeros would decrease and the maximum values increase. For positive α_2 the opposite would be the case.

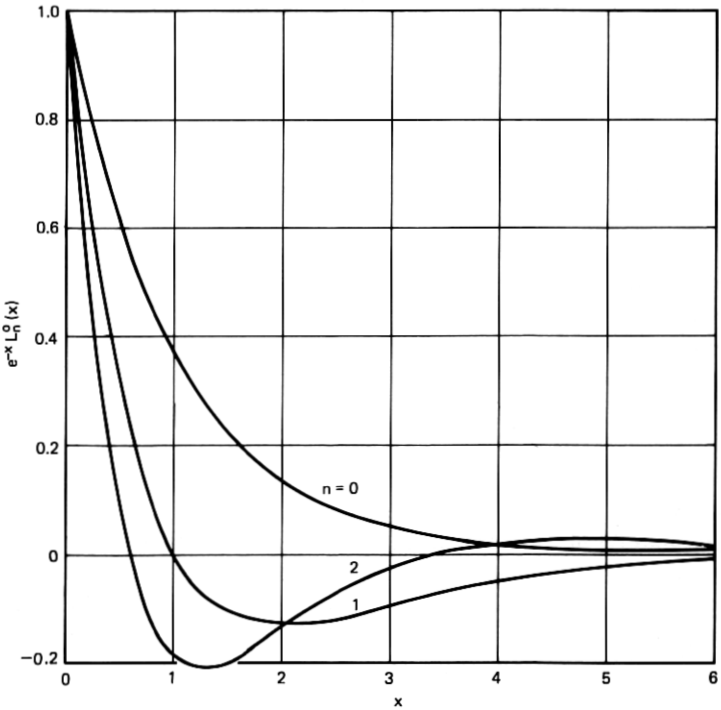

Fig. 3—Laguerre functions, $e^{-x}L_n^0(x)$.

It has been shown above that the transient response of notch filters can be obtained from the Hankel transform of the low-pass impulse response. This property can be used to deduce the qualitative characteristics of notch filters derived from conventional low-pass filters of the Bessel, Butterworth, and Tschebyscheff type. The impulse response of conventional filters, with transfer function polynomials of the same order and approximately the same bandwidth, has similarities to the impulse response considered. Therefore, the graphs shown in Fig. 3 are also representative of the transient response of notch filters derived from conventional filters.

The transient response of notch filters can be kept arbitrarily low at large times, such times being defined after the first zero of the envelope, t_o. The time t_o can be kept small by the choice of the number of sections m, and/or by choosing β/α_1 large. However, for the interval $0 \leqq t \leqq t_o$ these methods are not effective. In fact, it has been shown above from the initial value theorem (27) that, at $t = 0$, the envelope of the response is unity independent of the filter parameters. A low-pass filter combined with a notch filter would cause the transient response to be zero at $t = 0$, and behave, for small t, as t^{k-1} if k is the order with which the transfer function goes to zero as $s \to \infty$. However, with a given low-pass filter the transient response may not be reduced to a desired level at small t. An additional method of reducing the response by the use of phasing sections is discussed subsequently.

4.4 Numerical computations

To illustrate some of the properties of notch filters, numerical computations for a three-section filter ($m = 2$), with the parameters given in Table I, have been performed. Figure 4 shows the filter response to a step function and Fig. 5 the response to a stepped cosine function. The computed results are essentially in agreement with those obtained based on the approximate method. Figure 6 shows the transient response of this filter followed by a C-message weighting filter, when excited with a stepped cosine function. A comparison of Figs. 5 and 6 shows that the C-message filter reduced, as expected, the first lobe of the response, affected only slightly the second lobe and increased the subsequent lobes. Increasing β and hence the 3-dB bandwidth of the notch is not very effective in reducing the second lobe. This led to the investigation of phasing sections as a means of reducing the transient response.

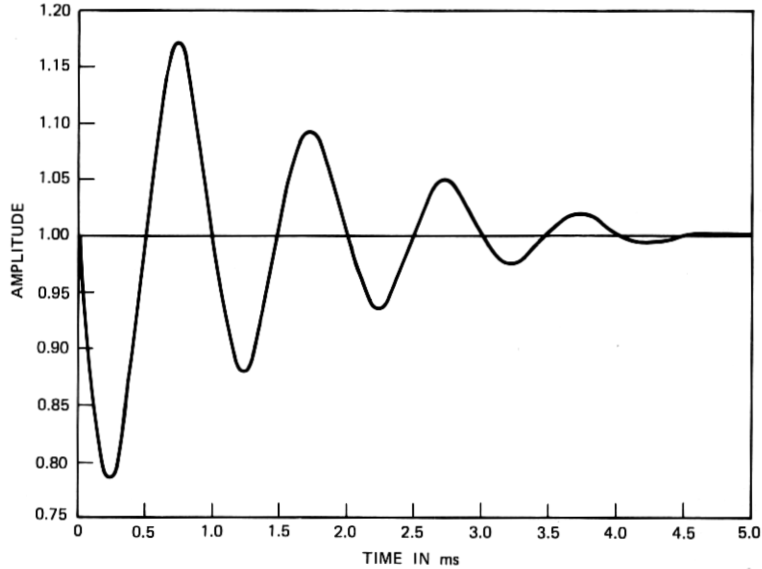

Fig. 4—Unit step response of notch filter.

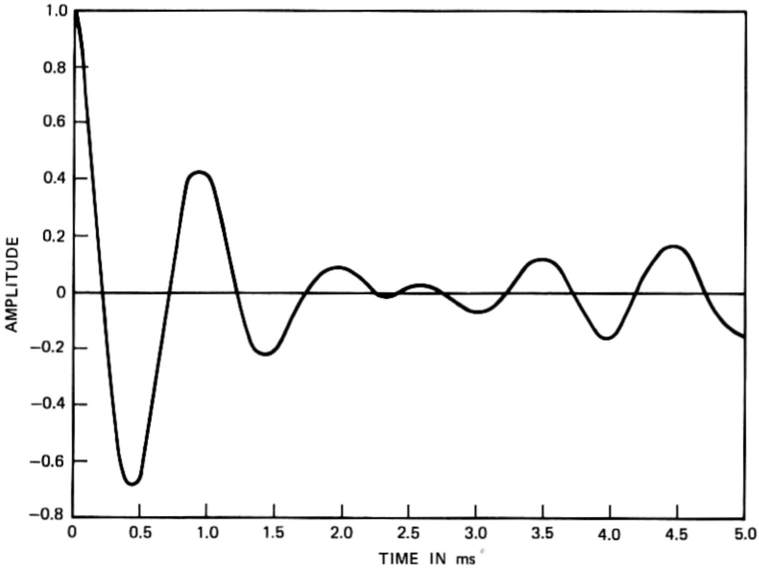

Fig. 5—Stepped cosine response of notch filter.

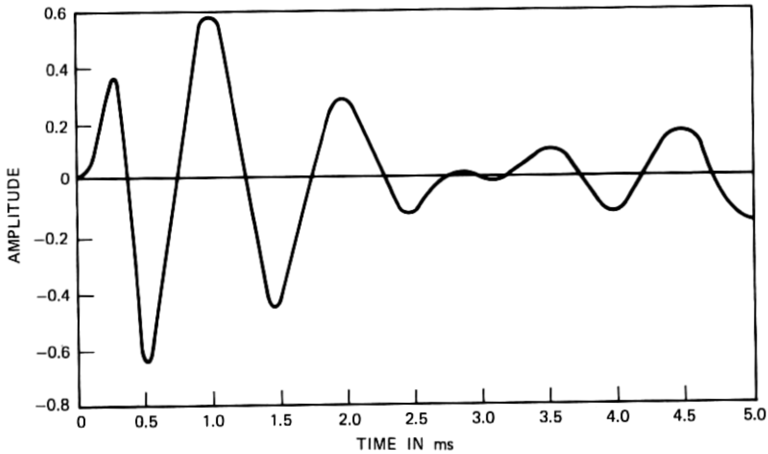

Fig. 6—Response of notch and C-message weighting filters to stepped cosine function.

V. TRANSIENT RESPONSE OF PHASING SECTIONS

The transient response of a phasing section is considered when excited by a damped sine function representing the output of the notched low-pass filter combination. Let the transfer function of the phasing section, $T_p(s)$, be given by

$$T_p(s) = \frac{s^2 - cs + d^2}{s^2 + cs + d^2} \tag{28}$$

and the Laplace transform of the damped sine function, $F_d(s)$, by

$$F_d(s) = \frac{\omega_d}{s^2 + as + b^2} \tag{29}$$

with $\omega_d = \sqrt{b^2 - (a/2)^2}$. The Laplace transform of the response, $F(s)$, is

$$F(s) = \frac{\omega_d}{s^2 + as + b^2} \frac{(s^2 - cs + d^2)}{s^2 + cs + d^2}. \tag{30}$$

Equation (30) can also be written

$$F(s) = \frac{\omega_d}{s^2 + as + b^2} - 2c\omega_d \left[\frac{A\left(s + \frac{a}{2}\right) + B}{s^2 + as + b^2} + \frac{C\left(s + \frac{c}{2}\right) + D}{s^2 + cs + d^2} \right], \tag{31}$$

where the constants A, B, C, and D are obtained by comparing (30) and (31).

It can be readily shown that the time response corresponding to (31), $f(t)$, is

$$f(t) = e^{-(a/2)t}\left[\sin \omega_d t - \frac{2cb}{r} \sin (\omega_d t + \theta_o)\right] + \frac{2cd}{r}\frac{\omega_d}{\omega_1} e^{-(c/2)t} \sin (\omega_1 t + \theta_1), \quad (32)$$

where

$$r = \sqrt{(d^2 - b^2)^2 + (c-a)[b^2c - d^2a]}, \quad (33)$$

$$\tan \theta_0 = \frac{(d^2 - b^2)\omega_d}{b^2(c-a) - a/2(d^2 - b^2)}, \quad (34)$$

$$\tan \theta_1 = \frac{(d^2 - b^2)\omega_1}{d^2(c-a) - c/2(d^2 - b^2)}, \quad (35)$$

and

$$\omega_1 = \sqrt{d^2 - \left(\frac{c}{2}\right)^2}. \quad (36)$$

The first term in (32) is the damped sine function and the other two terms are introduced by the phasing section. In order that the last two terms be of significance, it is necessary that these terms be comparable to the first term. This will be the case for $d = b$, for which (32) reduces to

$$f(t) = e^{-(a/2)t}\frac{c+a}{a-c} \sin \omega_d t - \frac{2c}{a-c}\frac{\omega_d}{\omega_1} e^{-(c/2)t} \sin \omega_1 t. \quad (37)$$

If, in addition, $(a/2)^2 \ll b^2$ and $(c/2)^2 \ll d^2$, (37) simplifies further and, for $(a/2 - c/2)t \ll 1$, (37) is approximately given by

$$f(t) \approx e^{-a/2t} \sin bt(1 - ct). \quad (38)$$

Equation (38) contains the damped sine function but modified by the term $(1 - ct)$. This term can be used to introduce a zero in the time vicinity where the damped sine function assumes a maximum value.

To illustrate the above, a phasing section is introduced to modify the impulse response of a C-message weighting filter. The computed impulse response of the filter is shown in Fig. 7 and has relatively large values in the time vicinity of 0.4 ms. The impulse response modified by a phasing section is shown in Fig. 8. The phasing section parameters are $c = 2 \cdot 10^3$ and $d = 10^8$. These parameters have been chosen on the basis of the above analysis. It is evident the large values of the response have been reduced, but the modified response has

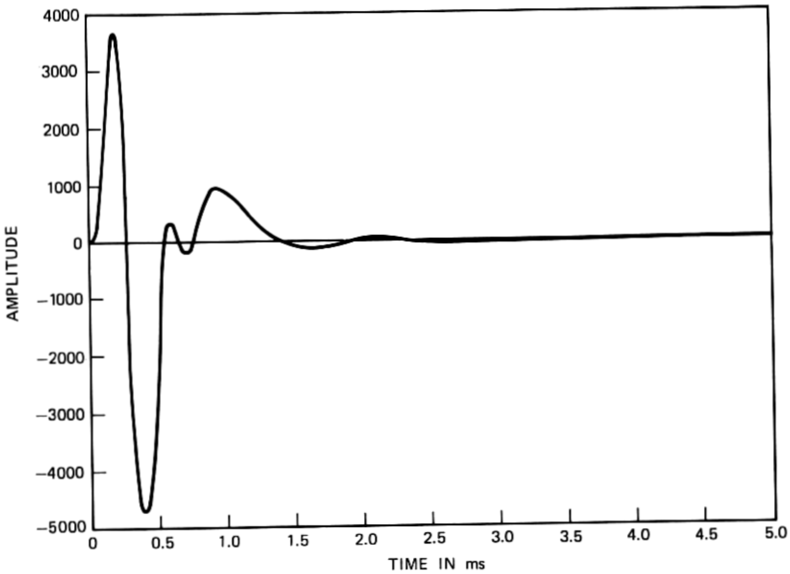

Fig. 7—Impulse response of C-message weighting filter.

appreciable values for a much longer time duration than the initial response. This behavior can be explained on the basis of Parseval's theorem,[14] since the absolute value of the Fourier transform of the response is the same with and without the phasing section.

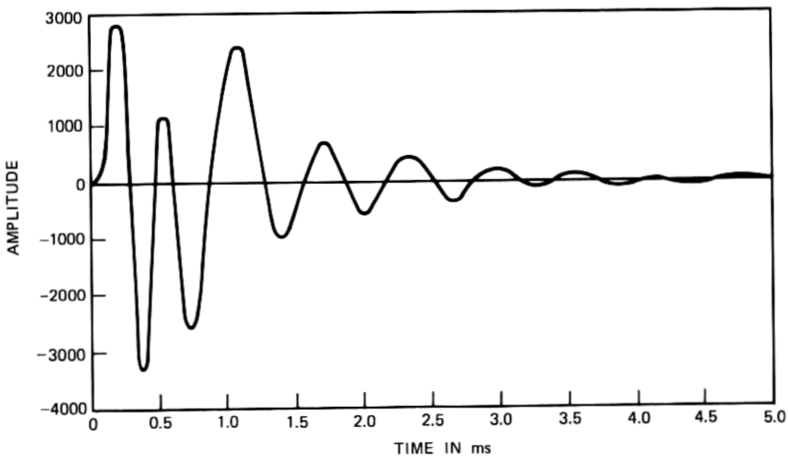

Fig. 8—Impulse response of C-message weighting filter with phasing section.

Table II—Two-section filter with phasing section (3-dB bandwidth 390 Hz, 30-dB bandwidth 80 Hz)

i	Parameters in kHz Units			
	σ_{ni}	ω_{ni}^2	σ_{di}	ω_{di}^2
1	0.09	40.27	1.35	27.91
2	0.09	40.27	1.94	51.10
3	−1.8	100.00	1.8	100.00

Phasing sections can be used to reduce the overshoot of the response of a notch filter followed by a low or bandpass filter and excited with a stepped trigonometric function at the notch frequency. Such sections are of particular importance where the transient response has to be modified without affecting the amplitude of the frequency response or where a modification is needed at times shortly after the beginning of the response. However, such sections may also introduce considerable distortions of the unit step response.

As an example, the performance of a 1010-Hz notch filter with and without a phasing section is considered. The transfer function, $T(z)$, of the filter and phasing section can be written as

$$T(z) = \prod_{i=1}^{3} \frac{z^2 + \sigma_{ni}z + \omega_{ni}^2}{z^2 + \sigma_{di}z + \omega_{di}^2}. \tag{39}$$

The parameters in (39) are listed in Table II.

This filter was derived from a Tschebyscheff low-pass filter, and an operational amplifier version was synthesized and built. Figures 9a and 9b show the computed response to a stepped cosine of the filter combined with a C-message weighting filter without and with the phasing section. Figures 9c and 9d show photos of the corresponding oscilloscope displays obtained with the actual filters. Good agreement was obtained between the computed and measured response. The effect of the phasing section on the response is evident in this figure. About a 4-dB reduction in the overshoot was obtained with the phasing section.

VI. CONCLUSIONS

The transient response of a class of notch filters which are derived from low-pass filters by a frequency transformation was investigated. General expressions for the transient response due to a unit step

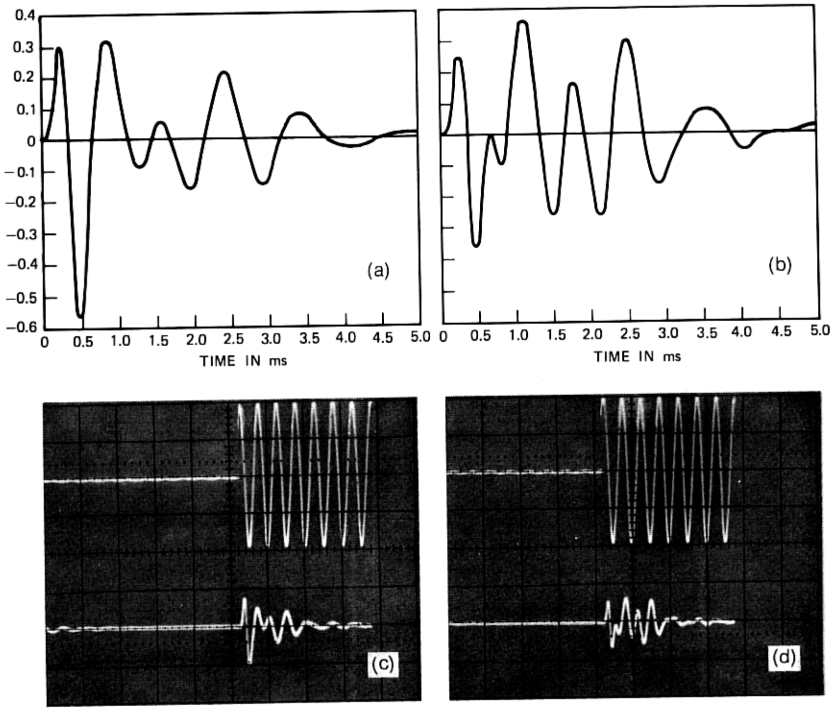

Fig. 9—Computed and measured response of notch filter with C-message weighting filter to a stepped cosine: (a) and (c) without phasing section, (b) and (d) with phasing section.

function and a stepped trigonometric function have been obtained in terms of the low-pass impulse response.

The transient response of certain types of notch filters can be formulated approximately in terms of Laguerre functions. These filters have been examined in detail and some general properties of the notch filter response have been deduced from this formulation.

Notch filters may considerably distort short time pulses (time duration less than one-half notch frequency period). The amount of distortion depends on the notch depth.

The response of notch filters to stepped trigonometric functions can be kept at low levels only after a certain time interval from the beginning of the response. The length of the time interval depends on the filter parameters.

A method of reducing the transient response at short time intervals by the use of phasing sections was presented. This method may prove particularly useful in applications where it is necessary to modify the transient response without affecting the frequency response.

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APPENDIX A

Transient Integrals

A.1 Unit step response

The relationship between the step response of the notch filter and the impulse response of the low-pass filter from which the notch filter is derived is given by (6) of the text,

$$f_{NS}(t) = \frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z}\int_0^\infty \exp\left(-\frac{\beta z}{z^2+\omega_o^2}u\right) f_L(u)du\,dz. \tag{40}$$

After interchanging the order of the integrations and expanding the exponential function, (40) can be written

$$f_{NS}(t) = \frac{1}{2\pi j}\int_0^\infty f_L(u)\int_\Gamma \frac{e^{zt}}{z}\sum_{n=0}^{\infty}\frac{(-\beta u)^n}{n!}\left(\frac{z}{z^2+\omega_o^2}\right)^n dz\,du. \tag{41}$$

Equation (41) contains a sum of inverse Laplace transforms of a tabulated form [15,16]

$$\frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z^{2v+1}} G\left(z+\frac{\omega_o^2}{z}\right)dz = \int_0^t \left(\frac{t-u}{\omega_o^2 u}\right)^v J_{2v}[2\omega_o\sqrt{ut-u^2}]g(u)du, \tag{42}$$

where $g(u)$ is the inverse Laplace transform of $G(z)$.

Equation (42) is of its own interest, since it may be used to obtain the step response of a bandpass filter derived from a low-pass filter by a frequency transformation. A direct derivation of (42) follows.

Using the definition of $G[z+(\omega_o^2)/z]$, (42) can be written

$$\frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z^{2v+1}} G\left(z+\frac{\omega_o^2}{z}\right)dz = \frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z^{2v+1}}\int_0^\infty \exp\left[-\left(z+\frac{\omega_o^2}{z}\right)u\right]g(u)du. \tag{43}$$

After interchanging the order of integration and expanding the exponential function in a power series, (43) can be written

$$\frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z^{2v+1}}G\left(z+\frac{\omega_o^2}{z}\right)dt = \frac{1}{2\pi j}\int_0^\infty g(u)\int_\Gamma e^{z(t-u)}\sum_{m=0}^{\infty}\frac{(-\omega_o^2 u)^m}{m!\,z^{m+2v+1}}\,dz\,du. \tag{44}$$

The inverse of each term in (44) is zero for a negative exponential argument. For a positive argument $u \leqq t$, the inverse is readily obtained; hence,

$$\frac{1}{2\pi j}\int_\Gamma \frac{e^{zt}}{z^{2v+1}}G\left(z+\frac{\omega_o^2}{z}\right)dz = \int_0^t g(u)\sum_{m=0}^{\infty}\frac{(-1)^m(\omega_o^2 u)^m}{m!}\,\frac{(t-u)^{m+2v}}{(m+2v)!}\,du. \tag{45}$$

The summation of the terms in (45) gives a Bessel function [17] of order $2v$, and hence (42). Using (42), (41) can be written

$$f_{NS}(t) = \int_0^\infty f_L(u)du + \int_0^\infty f_L(u)\int_0^t \sum_{n=1}^{\infty}\frac{(-\beta u)^n}{n!}\,\frac{x^{n-1}}{(n-1)!} \cdot J_o(2\omega_o\sqrt{xt-x^2})dx\,du. \tag{46}$$

The series in (46) can be summed yielding a Bessel function of order one; hence,

$$f_{NS}(t) = \int_0^\infty f_L(u)du - \int_0^t\int_0^\infty f_L(u)\sqrt{\frac{\beta u}{x}}\,J_1(2\sqrt{\beta x u})du \cdot J_0(2\omega_o\sqrt{tx-x^2})dx. \tag{47}$$

The integration with respect to u can be interpreted as a Hankel transform of the low-pass impulse response. For an impulse response which is not very oscillatory, and for $\beta \ll \omega_o$, the Hankel transform will be a slowly varying function in comparison to the Bessel function. Under these conditions an approximation to (47) can be obtained by using the method of stationary phase.

A.2 The stationary phase approximation

The stationary phase method[18] approximates integrals, I, of the following type:

$$I = \int_a^b g(x)e^{jk\psi(x)}dx, \tag{48}$$

where k is large and $g(x)$ is a slowly varying function. The approximation considers only contributions from the vicinity of stationary points where $(d\psi)/dx = 0$, and is of order $(1/k)$. The approximate value of the integral (48) is

$$I \approx \sum_i g(x_i) \sqrt{\frac{2j\pi}{k\psi''(x_i)}}\, e^{jk\psi(x_i)}. \tag{49}$$

To bring (47) to a form suitable for evaluation with the stationary phase method, the Bessel function is expressed in terms of modulus and phase.[19]

$$J_o(z) = M_o(z) \cos \theta_o(z) \tag{50}$$

with

$$\theta_o(z) = z - \frac{\pi}{4} + \delta_o(z), \tag{51}$$

where $\delta_o(z)$ and $M_o(z)$ are slowly varying functions for large z with $\lim_{z\to\infty} \delta_o(z) = 0$ and $\lim_{z\to\infty} M_o(z) = \sqrt{2/(\pi z)}$.

With $z = 2\omega_0\sqrt{tx - x^2}$, the integral in (47) has a stationary point at $x = t/2$. The approximate value of (47), obtained by using (49), is

$$f_{NS}(f) \approx \int_0^\infty f_L(u)dt - \sqrt{\frac{\pi t}{2\omega_0}}\, M_o(\omega_o t) \cos\left[\theta_o(\omega_o t) - \frac{\pi}{4}\right] \cdot \int_0^\infty f_L(u)\sqrt{\frac{2\beta u}{t}}\, J_1(\sqrt{2\beta ut})du. \tag{52}$$

For large values of t such that $M_o(z)$ and $\theta_o(t)$ can be approximated with their asymptotic values,

$$f_{NS}(t) \approx \int_0^\infty f_L(u)dt - \frac{\sin \omega_o t}{\omega_o} \int_0^\infty f_L(u)\sqrt{\frac{2\beta u}{t}}\, J_1(\sqrt{2\beta ut})du. \tag{53}$$

It is of interest to note that the stationary phase method gives the correct value for the integral

$$\int_0^t J_o(2\omega_o\sqrt{ut - u^2})du = \frac{\sin \omega_o t}{\omega_0}. \tag{54}$$

This integral can be evaluated exactly by using (42) with $v = 0$ and $g(u) = 1.0$. The left-hand side of (42) is readily inverted yielding (54).

APPENDIX B

Comparison of Exact and Approximate Solutions

Consider a notch filter transfer function

$$T_N(z) = \left(\frac{z^2 + \omega_o^2}{z^2 + \beta z + \omega_o^2}\right)^{m+1}. \tag{55}$$

The Laplace transform of the time response due to a unit step function is

$$F_{NS}(z) = \frac{1}{z}\left(1 - \frac{\beta z}{z^2 + \beta z + \omega_o^2}\right)^{m+1}. \tag{56}$$

For $m = 0$, the time response is

$$f_{NS}(t) = \left(1 - \frac{\beta}{\omega}\, e^{-(\beta/2)t} \sin \omega t\right) \cdot 1(t), \tag{57}$$

where

$$\omega = \sqrt{\omega_o^2 - \left(\frac{\beta}{2}\right)^2}. \tag{58}$$

A comparison of (57) with (19) shows that both are of the same form but ω is replaced by ω_o. Hence, the approximation is of order $(\beta/\omega_o)^2$.

For $m = 1$, using tables of Laplace transforms,

$$f_{NS}(t) = 1 - \frac{\beta}{\omega}\left[2 + \left(\frac{\beta}{2\omega}\right)^2 - \frac{\beta}{2}\,t\right] e^{-(\beta/2)t} \sin \omega t + \left(\frac{\beta}{\omega}\right)^2 \beta t e^{-(\beta/2)t} \cos \omega t. \tag{59}$$

Neglecting terms of order $(\beta/\omega)^2$ against one, (59) can be written

$$f_{NS}(t) = 1 - \frac{\beta}{\omega_o}\, e^{-(\beta/2)t} L_1^1\left(\frac{\beta}{2}\,t\right) \sin \omega_o t + \left(\frac{\beta}{2\omega_o}\right)^2 \beta t e^{-(\beta/2)t} \cos \omega_o t, \tag{60}$$

where from (16)

$$L_1^1(x) = 2 - x. \tag{61}$$

The terms multiplied by $\sin \omega_o t$ are up to order $(\beta/\omega_o)^2$ the same as in (19).

For $m = 2$,

$$\begin{aligned} f_{NS}(t) = 1 - \frac{\beta}{\omega}\, e^{-(\beta/2)t} \Bigg[& 3 + \frac{7}{8}\frac{\beta^2}{\omega^2} + \frac{3}{32}\left(\frac{\beta}{\omega}\right)^4 \\ & - \frac{\beta t}{2}\left(3 + \frac{\beta^2}{4\omega^2}\right) + \frac{1}{2}\left(\frac{\beta t}{2}\right)^2\left(1 - \frac{\beta^2}{4\omega^2}\right)\Bigg] \sin \omega t \\ & + \frac{\beta^2}{\omega^2}\, e^{-(\beta/2)t} \beta t \left[\frac{7}{8} + \frac{3}{32}\frac{\beta^2}{\omega^2} - \frac{\beta t}{8}\right] \cos \omega t. \end{aligned} \tag{62}$$

Neglecting terms of order $(\beta/\omega_o)^2$ and higher against one yields

$$f_{NS}(t) = 1 - \frac{\beta}{\omega_o} e^{-(\beta/2)t} L_2^1\left(\frac{\beta}{2}t\right) \sin \omega_o t + \frac{\beta^2}{\omega^2} e^{-(\beta/2)t} \frac{\beta t}{8} [7 - \beta t] \cos \omega_o t, \quad (63)$$

where from (16)

$$L_2^1(x) = 3 - 3x + \frac{x^2}{2}. \quad (64)$$

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